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Baroclinic instability of quasi-geostrophic flows localized in a thin layer

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This paper examines the baroclinic instability of a quasi-geostrophic flow with vertical shear in a continuously stratified fluid. The flow and density stratification are both localized in a thin upper layer. (i) Disturbances whose wavelength is much smaller than the deformation radius (based on the depth of the upper layer) are demonstrated to satisfy an 'equivalent two-layer model' with properly chosen parameters. (ii) For disturbances whose wavelength is of the order of, or greater than, the deformation radius we derive a sufficient stability criterion. The above analysis is applied to the subtropical and subarctic frontal currents in the Northern Pacific. The effective time of growth of disturbances (i) is found to be 16–22 days, the characteristic spatial scale is 130–150 km.

1. Introduction

Although the baroclinic instability of zonal currents is one of the classical problems of physical oceanography, a very broad class of 'solvable' flows seems to have been overlooked: flows localized in a thin layer. On the one hand, the assumption of localization can provide a basis for an asymptotic theory; on the other hand, most of the major oceanic currents (except the Gulf Stream and ACC) are indeed localized in a thin upper layer.

Another important oceanographic problem, which turns out to be relevant to this study, is the two-layer model. Can it be calibrated to quantitatively describe real-life (continuously stratified) flows? If it can, then is there any dynamical basis for this description, or is it just a result of skilful use of best fits?

The main result of this work is the asymptotic derivation of the two-layer model from the continuous model for flows localized in a thin layer. Attention will be focused on unbounded currents without horizontal shear. We shall consider three approximations of vertical structure of the flows:

(i) the three-layer model with two thin layers adjacent to the surface (§2);

(ii) the 'mixed' model, which consists of a thin continuously stratified layer on top of a thick homogeneous layer (§§3–5);

(iii) the continuous model with stratification and flow localized in a thin upper layer, i.e. the general case (Appendix A).

The results obtained theoretically are applied to the subtropical and subarctic frontal currents in the Northern Pacific (§6).

2. The three-layer model

In order to clarify the most robust features of the problem at hand, we shall first consider the (simplest) three-layer model. The bottom (thick) layer represents the below-thermocline (homogeneous) part of the ocean, while the two upper layers represent the so-called 'active layer', where the flow and density stratification are mostly localized. The main difference between the two- and three-layer models is that the latter has a 'profiled' active layer.

2.1. *Governing equations*

Consider a three-layer fluid on the β-plane. This system is characterized by the (dimensional) depths and densities of the layers: $\tilde{h}_j$ and $\tilde{\rho}_j (j = 1, 2, 3)$. We also introduce a set of global parameters: the total depth

$$H_0 = \tilde{h}_1 + \tilde{h}_2 + \tilde{h}_3,$$

the global density variation

$$\delta\rho = \tilde{\rho}_3 - \tilde{\rho}_1,$$

the surface density

$$\rho_s = \tilde{\rho}_1,$$

and the deformation radius based on H_0:

$$R_0 = (g' H_0)^{1/2} / f, \tag{2.1}$$

where $g' = (\delta\rho/\rho_s) g$ is the reduced acceleration due to gravity and f is the Coriolis parameter. Finally, we introduce the 'β-effect number':

$$\alpha = (R_0/R_e) \cot\theta, \tag{2.2}$$

where R_e is the Earth's radius and θ is the latitude. (α can be interpreted as the non-dimensional version of the usual β-parameter: $\alpha = R_0 \beta/f$. We shall use α instead of the usual β-effect number defined by $\beta L^2/U$ (e.g. Pedlosky 1987).)

We shall use a set of non-dimensional variables related to the dimensional spatial coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$, the time $\tilde{t}$, the pressure $\tilde{\Psi}$, and the density $\tilde{\rho}$ as follows:

$$t = \tilde{t} f, \quad x = \frac{\tilde{x}}{R_0}, \quad y = \frac{\tilde{y}}{R_0}, \quad \Psi_j = \frac{(\tilde{\Psi}_j - gz) f}{g' H_0}, \quad \rho_j = \frac{\tilde{\rho}_j - \rho_s}{\delta\rho}, \quad h_j = \frac{\tilde{h}_j}{H_0}. \tag{2.3}$$

In terms of these variables, the dynamics of a three-layer fluid is governed by the standard quasi-geostrophic equations:

$$\left.\begin{aligned} &\mathrm{D}_1\left[\nabla^2\Psi_1 - \frac{\Psi_1 - \Psi_2}{h_1(\rho_2 - \rho_1)} + \alpha y\right] = 0, \\ &\mathrm{D}_2\left[\nabla^2\Psi_2 - \frac{1}{h_2}\left(\frac{\Psi_2 - \Psi_1}{\rho_2 - \rho_1} - \frac{\Psi_3 - \Psi_2}{\rho_3 - \rho_2}\right) + \alpha y\right] = 0, \\ &\mathrm{D}_3\left[\nabla^2\Psi_3 - \frac{\Psi_3 - \Psi_2}{h_3(\rho_3 - \rho_2)} + \alpha y\right] = 0; \end{aligned}\right\} \tag{2.4}$$

where

$$\mathrm{D}_j \Phi = \frac{\partial\Phi}{\partial t} + J(\Psi_j, \Phi)$$

and

$$J(\Psi, \Phi) = \frac{\partial\Psi}{\partial x}\frac{\partial\Phi}{\partial y} - \frac{\partial\Psi}{\partial y}\frac{\partial\Phi}{\partial x}$$

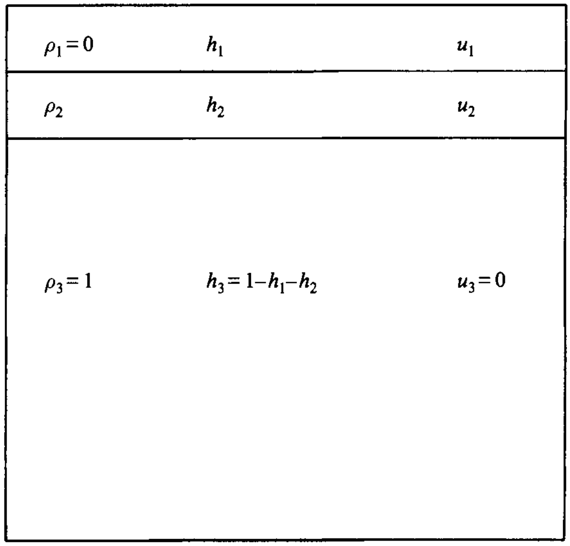

FIGURE 1. Three-layer model: non-dimensional formulation of the problem. ρ_j are the non-dimensional densities ($j = 1,2,3$), h_j are the non-dimensional depths of the layers, u_j are the non-dimensional velocities.

is the Jacobian operator. ρ_1, ρ_3 and h_3 can be eliminated from (2.4) by

$$\rho_1 = 0, \quad \rho_3 = 1, \quad h_3 = 1 - h_1 - h_2.$$

Then we linearize (2.4) against the background of a steady zonal flow without horizontal shear, i.e. substitute

$$\Psi_j = -u_j y + \psi_j$$

into (2.4) and omit the nonlinear terms. Without any loss of generality, we can assume that the bottom layer is at rest:

$$u_3 = 0$$

(all non-dimensional parameters of the three-layer model are shown in figure 1). Next we substitute the harmonic-wave solution

$$\psi_j = A_j \exp\left[\mathrm{i}m(ct - x) - \mathrm{i}ly\right]$$

((m, l) and c are the wavevector and phase speed, respectively), eliminate the constants A_j and obtain the dispersion relation:

$$(c - u_2)\left[h_1 \rho_2 (1-\rho_2) k^2 - \frac{(1-\rho_2)(c-u_1)}{(c-u_1) h_1 \rho_2 k^2 + c + h_1 \rho_2 \alpha - u_2} - \frac{\rho_2 c}{c(1-h_1-h_2)(1-\rho_2)k^2 + c + (1-h_1-h_2)(1-\rho_2)\alpha - u_2}\right] + c + h_2 \rho_2 (1-\rho_2)\alpha - u_1(1-\rho_2) = 0, \quad (2.5)$$

where $k^2 = m^2 + l^2$. This cubic equation (with respect to c) can be readily solved. However, owing to the large number of parameters involved, the exact solution is very bulky and meaningless from a physical viewpoint. We shall analyse (2.5) using the assumption that the two upper layers are thin:

$$\epsilon = H_a / H_0 \ll 1,$$

where $H_a = \tilde{h}_1 + \tilde{h}_2$ is the (dimensional) depth of the 'active' layer. Accordingly, we scale the depths of the upper layers as follows:

$$h_{1,2} = \epsilon \hat{h}_{1,2}. \tag{2.6a}$$

It should be recalled here that α and u_j are also small. Indeed, the smallness of α follows from (2.2) ($R_0 \ll R_e$), whereas the non-dimensional velocity u_j is scaled by $(g' H_0)^{1/2}$ (see (2.3)) and, in the case of geostrophic motion, is also small. Thus, α and u_j are to be scaled. The simplest and most natural scaling is

$$\alpha = \epsilon \hat{\alpha}, \quad u_{1,2} = \epsilon \hat{u}_{1,2}. \tag{2.6b}$$

Equation (2.6) can be rewritten as

$$\alpha \sim H_a/H_0, \quad Ro \sim H_a/H_0,$$

where Ro is the Rossby number. This condition corresponds to the 'regime of strong β-effect and thin upper layer' (see Benilov & Reznik 1995) and can be applied to a number of frontal flows in the Northern Pacific (we shall return to this question in §7). It should also be emphasized that, in contrast to condition (2.6*a*) (localization of the flow), the specific form of conditions (2.6*b*) is not crucial: for example, we could replace them by

$$\alpha = \epsilon^{3/2} \hat{\alpha}, \quad u_j = \epsilon \hat{u}_j,$$

which would correspond to the 'regime of weak β-effect and thin upper layer':

$$\alpha \sim (H_a/H_0)^{3/2}, \quad Ro \sim H_a/H_0$$

(see Benilov & Reznik 1995). In fact, we could use any pair of conditions which guarantee the smallness of $u_{1,2}$ and α.

We should also specify the asymptotic range for the wavenumber k. There are three asymptotic zones:

$$\text{long disturbances:} \quad k^2 \lesssim 1, \tag{2.7a}$$

$$\text{medium disturbances:} \quad k^2 \sim \epsilon^{-1/2}, \tag{2.7b}$$

$$\text{short disturbances:} \quad k^2 \gtrsim \epsilon^{-1}; \tag{2.7c}$$

where the dispersion curves of (2.5) behave in three different ways. For reasons elaborated in the end of this section, we shall consider only medium disturbances (2.7*b*). Correspondingly,

$$k = \epsilon^{1/4} \hat{k}. \tag{2.8}$$

Substituting (2.6) and (2.8) into the dispersion equation (2.5), we omit hats and calculate, to the leading order, the coefficients of c^3, c^2, etc. After straightforward, but cumbersome, calculations we obtain

$$k^2 c^3 - \epsilon u_2 k^2 c^2 + \epsilon^{5/2}[(1-\rho_2) E k^4 - u_2 \alpha] c + \epsilon^4 E k^4 [(1-\rho_2)\alpha - u_2] = 0, \tag{2.9}$$

where
$$E = h_1 u_1^2 + h_2 u_2^2.$$

The three roots of this equation can be estimated as follows:

$$c_{1,2} = O(\epsilon^{3/2}), \quad c_3 = O(\epsilon). \tag{2.10a, b}$$

In order to find $c_{1,2}$, we should scale c in accordance with (2.10*a*):

$$c = \epsilon^{3/2} \hat{c}. \tag{2.11}$$

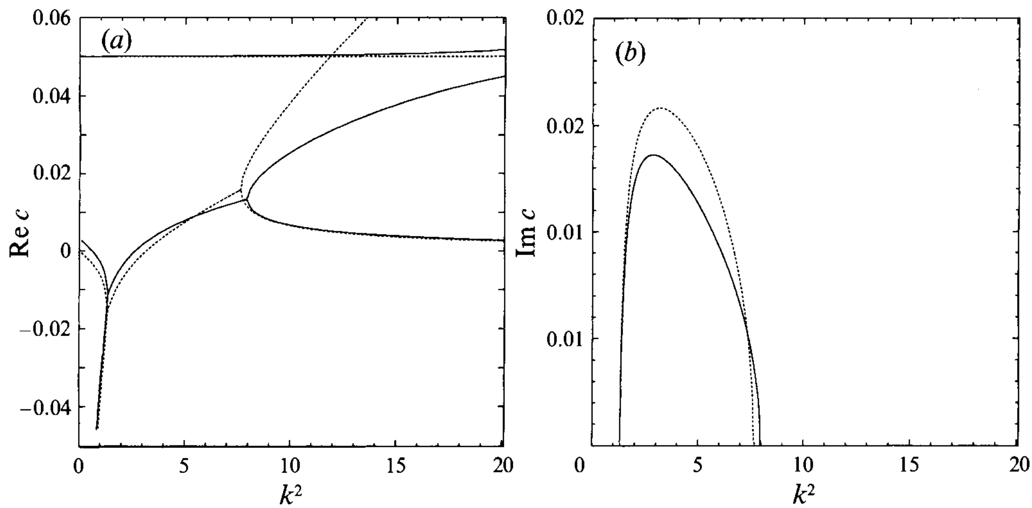

FIGURE 2. Phase speed (*a*) and growth rate (*b*) of medium disturbances in a three-layer eastward flow: $\rho_1 = 0$, $\rho_2 = 0.5$, $\rho_3 = 1$, $h_1 = 0.04$, $h_2 = 0.04$, $h_3 = 0.92$, $u_1 = 0.1$, $u_2 = 0.05$, $u_3 = 0$. The solid line shows the numerical solution of the exact dispersion equation (2.5), dashed line shows the asymptotic solution (2.12), (2.16). The region of instability is bounded by the points where the dispersion curves of modes 1 and 2 coalesce.

Then, substituting (2.11) into (2.9), we omit hats and small terms. The solution to the resulting (quadratic) equation is

$$c_1 = \frac{\bar{h}\bar{u}k^4 - \alpha - [(\bar{h}\bar{u})^2 k^8 + 2(\alpha - 2\bar{u})\,\bar{h}\bar{u}k^4 + \alpha^2]^{1/2}}{2k^2}, \tag{2.12a}$$

$$c_2 = \frac{\bar{h}\bar{u}k^4 - \alpha + [(\bar{h}\bar{u})^2 k^8 + 2(\alpha - 2\bar{u})\,\bar{h}\bar{u}k^4 + \alpha^2]^{1/2}}{2k^2}, \tag{2.12b}$$

where
$$\bar{u} = u_2/(1-\rho_2) \tag{2.13a}$$
and
$$\bar{h} = (1-\rho_2)^2 E/u_2^2 \tag{2.13b}$$

can be interpreted as the effective velocity and depth of the active layer, respectively.

It can be readily derived from (2.12) that $c_{1,2}$ is real for all k (stability) if and only if

$$0 \leqslant \bar{u} \leqslant \alpha. \tag{2.14}$$

Surprisingly, this stability criterion does not depend on $\bar{h}$. The latter parameter affects only the maximum growth rate and spectral boundaries of the baroclinic instability (if that occurs).

In order to find c_3, we scale c in accordance with (2.10*b*):

$$c = \epsilon\hat{c}. \tag{2.15}$$

Then, substituting (2.15) into (2.9), we omit hats and small terms:

$$c_3 = u_2. \tag{2.16}$$

This mode is always stable.

The approximate dispersion relations (2.12) and (2.16) are compared to the exact solution (obtained numerically) in figures 2 and 3.

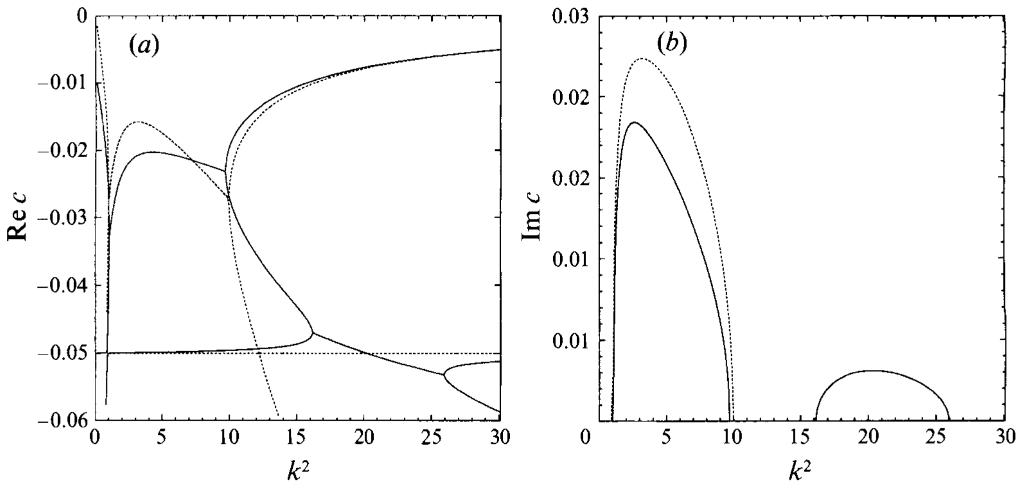

FIGURE 3. As figure 2 but for the westward flow with the same parameters. The main region of instability is bounded by the points where the dispersion curves of modes 1 and 2 coalesce. There is one more region of instability which corresponds to coalescence of modes 2 and 3. Observe that modes 2 and 3 do not coalesce in the left bottom corner of (*a*).

2.2. *Discussion*

(i) In order to illustrate the significance of (2.12), we shall recall the dispersion relation of the two-layer model with the depth of the upper layer $\epsilon\bar{h}$ and shear velocity $\epsilon\bar{u}$ (see Pedlosky 1987):

$$c_{1,2} = \frac{\bar{h}(1-\epsilon\bar{h})\,\bar{u}k^4 - \epsilon^{1/2}2\bar{h}[\bar{u}-(1-\epsilon\bar{h})\,\alpha]\,k^2+\alpha}{k^2+\epsilon^{1/2}\bar{h}(1-\epsilon\bar{h})\,k^4} \pm \frac{[(\bar{u}\bar{h})^2(1-\epsilon\bar{h})^2k^8+2(1-\epsilon\bar{h})[(1-2\epsilon\bar{h})\,\alpha-2\bar{u}]\,\bar{u}\bar{h}k^4+\alpha^2]^{1/2}}{k^2+\epsilon^{1/2}\bar{h}(1-\epsilon\bar{h})\,k^4}. \quad (2.17)$$

Then, upon taking the limit $\epsilon \rightarrow 0$, we see that (2.17) is (asymptotically) equivalent to (2.12). In other words,

the first two modes of the three-layer model with two thin upper layers are described by the two-layer model!
The parameters of the latter are related to the parameters of the former by (2.13).
The two-layer approximation is valid in the spectral range (2.7*b*).

It is also worth noting that the stability criterion (2.14) follows from the corresponding two-layer condition:

$$-\alpha\epsilon\bar{h} \leqslant \bar{u} \leqslant \alpha(1-\epsilon\bar{h})$$

as $\epsilon \rightarrow 0$.

(ii) It is convenient to rewrite spectral ranges (2.7) in the dimensional form:

$$\text{long disturbances:} \quad \lambda^2 \gtrsim R_0^2, \quad (2.18a)$$

$$\text{medium disturbances:} \quad R_a^2 \ll \lambda^2 \ll R_0^2, \quad (2.18b)$$

$$\text{short disturbances:} \quad \lambda^2 \lesssim R_a^2, \quad (2.18c)$$

where λ is the wavelength of a disturbance, R_0 is the deformation radius based on the total depth of the fluid (see (2.1)), and R_a is that based on the depth of the active layer:

$$R_a = (g'H_a)^{1/2}/f. \quad (2.19)$$

Clearly, for localized flows $R_a^2 \ll R_0^2$.

(iii) It is worth noting that the limit

$$u_2 \to 0, \quad u_1 \to \text{const} \neq 0$$

entails $\bar{h} \to \infty$, (see (2.13*b*)) and is inconsistent with the approximation of a thin upper layer. At the same time, the limit $u_1 \to 0$ does not violate the applicability of our results. In other words, *if we (incorrectly) include into the active layer a sublayer of still water, our approximation does not hold.* This conclusion also applies to the case of continuously stratified flows.

(iv) It is allowable, however, to take the double limit

$$u_1 \to 0, \quad u_2 \to 0$$

(waves in still water). The dispersion relations (2.12), (2.16) yield

$$c_1 = -\alpha/k^2, \quad c_{2,3} = 0.$$

These equalities demonstrate that c_1 represents the barotropic mode, whereas $c_{2,3}$ exist owing to the shear flow in the upper layer. Normally, a three-layer model has one barotropic and two baroclinic modes, which leaves us with a question: where are the baroclinic modes?

In order to answer this question, we estimate the phase speeds of the barotropic and baroclinic modes in still water:

$$c_1 = -\alpha/k^2, \quad c_{2,3} \sim \alpha/(k^2+h^{-1}),$$

where $h = h_1 + h_2$ is the non-dimensional depth of the active layer. Assuming that $\alpha \sim \epsilon$, $h_j \sim \epsilon$, $k^2 \sim \epsilon^{-1/2}$ (see (2.6), (2.7*b*)), we have

$$c_1 \sim \epsilon^{3/2}, \quad c_{2,3} \sim \epsilon^2 \quad \Rightarrow |c_1| \gg |c_{2,3}|.$$

Hence, baroclinic effects in a fluid with thin active layer are too weak (slow) to be taken into account. If, however, there is a sufficiently strong vertically sheared flow, the two higher modes are 'taken over' by the shear effects, which increase their phase speeds $c_{1,2}$.

(v) It should be noted that the accuracy of the asymptotic dispersion relation (2.12) is $O(\epsilon^{1/2})$ (see figures 2 and 3). In order to improve the agreement between the exact and asymptotic results, it is necessary to take into account the next correction to the phase speed.

(vi) From a mathematical point of view, the case of long disturbances (2.7*a*) can be treated similarly to that of medium disturbances. It can be demonstrated, however, that the original equations (2.4) in this case are not applicable to the real ocean. Indeed, estimating the slope of the interface between the active and passive layers:

$$s = -\bar{u},$$

one can find the 'outcropping distance' of the mean flow, i.e. the distance over which the interface would outcrop onto the surface of the ocean:

$$L = |\bar{h}/s| = (1-\rho_2)^3 E/|u_2|^3.$$

Estimating $u_2 \sim \epsilon$, $E \sim \epsilon^3$, we see that $L \sim 1$. Thus, for long disturbances the displacement of the interface is large ($kL \sim 1$), and the quasi-geostrophic equations (2.4) are not applicable.

Long disturbances should be studied using the primitive equations or the approximation of large-amplitude geostrophic flows (Benilov 1995).

(vii) Unlike the case of long disturbances, short disturbances (2.7 *c*) can be studied using the quasi-geostrophic approach. We shall not, however, go into the details, as the three-layer model cannot be reduced, in this case, to the two-layer model. Moreover, it turns out that the 'layer results' for the short-wave part of the spectrum do not have any correspondence with those for continuous stratification (the reason for which is the absence of critical levels in layer models).

As the three-layer model has no value for us in itself, we shall consider short disturbances in detail only for continuous stratification (§5). The only three-layer feature worth observing is the possibility of an additional zone of (weaker) instability in the short-wave region (see figure 3).

3. Mixed model: formulation of the problem

Our next step towards the general case (of continuous stratification) is the 'mixed' model, which consists of a thin continuously stratified layer on top of a thick homogeneous layer (figure 4). On the one hand, the mixed model is a reasonable approximation of the (continuously stratified) real ocean; on the other hand, it is not as complex as the fully continuous model. The latter does not add anything substantial to our understanding of the problem at hand and is considered in Appendix A.

In this case, the density stratification is described by

$$\rho(z) = \begin{cases} \rho(z) & \text{for} \quad z > -h, \\ 1 & \text{for} \quad z \leqslant -h, \end{cases}$$

where $\rho = (\tilde{\rho} - \rho_s)/\delta\rho$, $\tilde{\rho}(z)$ is the dimensional density, $\rho_s = \tilde{\rho}(0)$, $\delta\rho = \tilde{\rho}(-H_0) - \tilde{\rho}(0)$, z is the non-dimensional vertical variable (scaled by the total depth of the fluid H_0), and h is the non-dimensional depth of the upper (active) layer (also scaled by H_0).

3.1. *Governing equations*

The (non-dimensional) streamfunction is now a function of three spatial variables $\Psi(t, x, y, z)$. Otherwise, we shall use the same notation as in the previous case.

The standard quasi-geostrophic equation for Ψ is

$$\left[\nabla^2\Psi - \left(\frac{1}{\rho_z}\Psi_z\right)_z\right]_t + J\left[\Psi, \nabla^2\Psi - \left(\frac{1}{\rho_z}\Psi_z\right)_z\right] + \alpha\Psi_x = 0. \tag{3.1}$$

The no-flow conditions at the (rigid) boundaries are

$$\left(\frac{1}{\rho_z}\Psi_z\right)_t + J\left(\Psi, \frac{1}{\rho_z}\Psi_z\right) = 0 \quad \text{at} \quad z = 0, \tag{3.2}$$

$$\left(\frac{1}{\rho_z}\Psi_z\right)_t + J\left(\Psi, \frac{1}{\rho_z}\Psi_z\right) = 0 \quad \text{at} \quad z = -1. \tag{3.3}$$

In order to regularize the singularity in equation (3.1) ($1/\rho_z \equiv \infty$ for $z \leqslant -h$), one should consider 'almost' constant density: $|\rho_z| \ll 1$, and seek a solution in the form of a series in powers of ρ_z:

$$\Psi(t, x, y, z) = P(t, x, y) + Q(t, x, y)\int_{-1}^{z}(1 + z')\,\rho_{z'}(z')\,\mathrm{d}z' + O(\rho_z^2) \quad \text{for} \quad z \in (-1, -h) \tag{3.4}$$

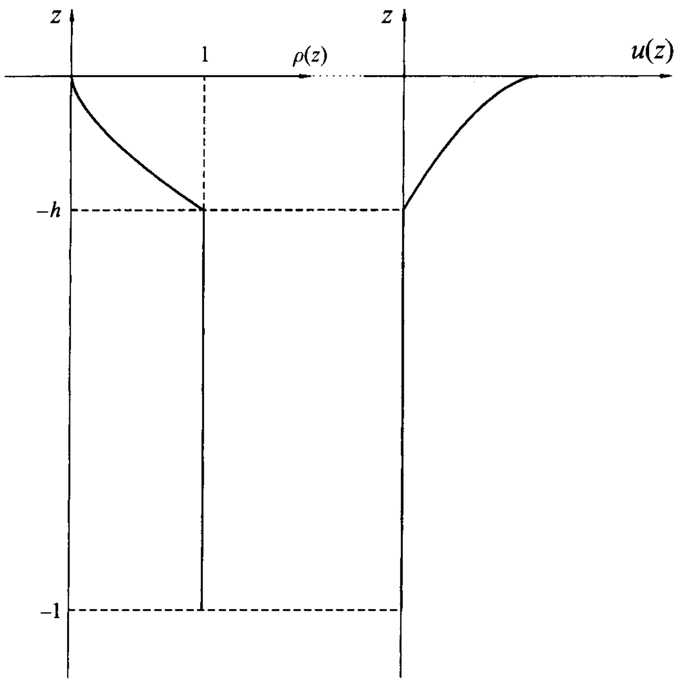

FIGURE 4. Mixed model: non-dimensional formulation of the problem: z is the non-dimensional vertical spatial variable, h is the non-dimensional depth of the active layer, $\rho(z)$ is the dimensional density, u is the non-dimensional velocity.

(this expansion automatically satisfies the bottom boundary condition (3.3)). Substituting (3.4) into (3.1) and taking the limit $\rho_z \to 0$, we obtain

$$(\nabla^2 P - Q)_t + J(P, \nabla^2 P - Q) + \alpha P_x = 0. \tag{3.5}$$

From a physical viewpoint, P is the pressure in the lower layer and Q is the displacement of the interface scaled by $(1-h)$.

In order to derive the matching condition at $z = -h$, we observe that the pressure and vertical velocity must be continuous:

$$(\Psi)_{z=-h+0} = (\Psi)_{z=-h-0}, \quad \left(\frac{1}{\rho_z}\Psi_z\right)_{z=-h+0} = \left(\frac{1}{\rho_z}\Psi_z\right)_{z=-h-0}. \tag{3.6}$$

Substituting (3.4) into (3.6) and taking the limit $\rho_z \to 0$, we obtain

$$(\Psi)_{z=-h+0} = P, \quad \left(\frac{1}{\rho_z}\Psi_z\right)_{z=-h+0} = (1-h)\,Q. \tag{3.7}$$

The matching conditions (3.7) are natural in the sense that they follow directly from the differential equation.

Equations (3.7), (3.5) and (3.2) supply boundary conditions for equation (3.1), which is now to be solved in the interval $z \in (-h, 0)$.

As before, we consider a flow without horizontal shear with the lower layer being at rest:

$$\Psi = -yu(z), \quad P = 0, \quad Q = ys, \tag{3.8a}$$

where the velocity $u(z)$ and slope of the interface s must satisfy the following constraints:

$$u = 0 \quad \text{at} \quad z = -h, \tag{3.8b}$$

$$\frac{1}{\rho_z} u_z = s \quad \text{at} \quad z = -h \tag{3.8c}$$

((3.8 b, c) were obtained by substitution of (3.8 a) into (3.7)). Linearizing the governing equation against the background of (3.8), we seek a harmonic-wave solution:

$$\Psi(t,x,y,z) = -yu(z) + \psi(z)\exp[\mathrm{i}m(ct-x) - \mathrm{i}ly],$$
$$P(t,x,y) = P\exp[\mathrm{i}m(ct-x) - \mathrm{i}ly],$$
$$Q(t,x,y) = -ys + Q\exp[\mathrm{i}m(ct-x) - \mathrm{i}ly].$$

Substituting these equalities into (3.1), (3.2), (3.5) and (3.7), and omitting nonlinear terms, we obtain

$$(c-u)\left[k^2\psi + \left(\frac{1}{\rho_z}\psi_z\right)_z\right] + \left[\alpha + \left(\frac{1}{\rho_z}u_z\right)_z\right]\psi = 0, \tag{3.9a}$$

$$\left.\begin{aligned} &(c-u)\,\psi_z = -u_z\,\psi && \text{at} \quad z = 0, \\ &c\frac{1}{\rho_z}\psi_z = -[(1-h)(ck^2+\alpha)+s]\,\psi && \text{at} \quad z = -1 \end{aligned}\right\} \tag{3.9b}$$

It is convenient to change the variables as follows:

$$\psi = (c-u)\,\phi, \quad z = h\xi. \tag{3.10}$$

Substituting (3.10) into (3.9) and taking into account (3.8 b, c), we obtain

$$\left[(c-u)^2\frac{1}{\rho_\xi}\phi_\xi\right]_\xi + h[k^2(c-u)^2 + \alpha(c-u)]\,\phi = 0, \tag{3.11a}$$

$$\left.\begin{aligned} &\phi_\xi = 0 && \text{at} \quad \xi = 0, \\ &c\frac{1}{\rho_\xi}\phi_\xi = -(1-h)(ck^2+\alpha)\,\phi && \text{at} \quad \xi = -1; \end{aligned}\right\} \tag{3.11b}$$

c and ϕ are the eigenvalue and eigenfunction of the boundary value problem (3.11). If $\operatorname{Im} c < 0$, the flow with parameters h, $u(\xi)$ and $\rho(\xi)$ is unstable. We shall also derive a useful integral identity: integrating (3.11 a) with respect to ξ over $(-1, 0)$ and taking into account the boundary conditions (3.11 b), we obtain

$$(1-h)\,c(ck^2+\alpha)\,\phi(-1) + h\int_{-1}^{0}[k^2(c-u)^2 + \alpha(c-u)]\,\phi\,\mathrm{d}\xi = 0. \tag{3.12}$$

Everywhere in this paper we shall assume that

$$\rho_\xi(\xi) < 0 \quad \text{for} \quad \xi \in (-1, 0),$$

which means that the stratification is stable in the static sense. We shall also assume that

$$u_\xi(-1) \neq 0,$$

which reflects the fact that the active layer, by definition, must have an order-one shear at the interface (otherwise the passive layer could have been extended). This restriction is similar to the condition $u_2 \neq 0$ for the three-layer model (see §2.2(iii)).

3.2. *The analogue of Rayleigh's stability criterion*

In order to derive a sufficient stability condition similar to Rayleigh's criterion for Couette flow (e.g. Dikiy 1976), we return to (3.9*a*), multiply it by $\psi^*/(c-u)$ and integrate with respect to z over $(-h, 0)$. Integrating by parts, using the boundary conditions (3.9*b*) and taking the imaginary part, we obtain

$$(\operatorname{Im} c)\left[-\frac{\frac{1}{\rho_z(0)}u_z(0)}{|c-u(0)|^2}|\psi(0)|^2+\frac{(1-h)\alpha+s}{|c|^2}|\psi(-h)|^2+\int_{-h}^{0}\frac{\alpha+\left(\frac{1}{\rho_z}u_z\right)_z}{|c-u|^2}|\psi|^2\,\mathrm{d}z\right]=0. \tag{3.13}$$

Equation (3.13) yields the following sufficient condition restricting the potential vorticity profile:

$$\left.\begin{array}{l}\alpha+\left(\dfrac{1}{\rho_z}u_z\right)_z \text{ does not change sign on } (-h,0),\\ u_z(0) \text{ has the same sign as } \alpha+\left(\dfrac{1}{\rho_z}u_z\right)_z\end{array}\right\} \tag{3.14}$$

and

$$(1-h)\alpha+s \text{ has the same sign as } \alpha+\left(\frac{1}{\rho_z}u_z\right)_z. \tag{3.15}$$

Condition (3.15) can be rewritten using the mean-value theorem. Indeed, given that

$$\left[\frac{1}{\rho_z}u_z\right]_{z=0}=\frac{1}{\rho_z(0)}u_z(0),\quad \left[\frac{1}{\rho_z}u_z\right]_{z=-h}=s$$

(see (3.8*c*)), there is a point z_0 such that

$$\left[\left(\frac{1}{\rho_z}u_z\right)_z\right]_{z=z_0}=\frac{\frac{1}{\rho_z(0)}u_z(0)-s}{h}. \tag{3.16}$$

As $\alpha+((1/\rho_z)u_z)_z$ does not change sign, we can replace (3.15) by

$$[(1-h)\alpha+s]\left[\alpha+\left(\frac{1}{\rho_z}u_z\right)_z\right]_{z=z_0}\geqslant 0. \tag{3.17}$$

Substitution of (3.16) into (3.17) yields

$$-(1-h)\alpha\leqslant s\leqslant h\left[\alpha+\frac{1}{\rho_z(0)}u_z(0)\right]. \tag{3.18}$$

It should be recalled here that s is the slope of the interface between the passive and active layers.

Equations (3.14) and (3.18) form a sufficient criterion of stability. In the general case, it is impossible to tell which one of the two conditions guarantees the stability of which disturbances (long, medium or short). In the case of localized flows, however, it will be demonstrated that (3.18) guarantees the stability of medium disturbances, whereas (3.14) guarantees the stability of short disturbances. Moreover, (3.18) turns out to be a necessary condition as well: the flows that do not satisfy it are certainly unstable.

4. Mixed model: medium disturbances

As before, we shall consider the regime of strong β-effect and thin upper layer, within the framework of which h, u and α should be scaled as follows:

$$h = \epsilon\hat{h}, \quad u = \epsilon\hat{u}, \quad \alpha = \epsilon\hat{\alpha}, \tag{4.1}$$

where ϵ is a small parameter (equal to the Rossby number); k and c are to be scaled exactly as they were in the case of the three-layer model and medium disturbances (see (2.8) and (2.11)):

$$k = \epsilon^{-1/4}\hat{k}, \quad c = \epsilon^{3/2}\hat{c}. \tag{4.2}$$

Substitution of (4.1)–(4.2) into (3.11) yields (hats omitted):

$$\left[(\epsilon^{1/2}c-u)^2\frac{1}{\rho_\xi}\phi_\xi\right]_\xi + h[\epsilon^{1/2}k^2(\epsilon^{1/2}c-u)^2+\epsilon\alpha(\epsilon^{1/2}c-\epsilon u)]\phi = 0, \tag{4.3a}$$

$$\phi_\xi = 0 \qquad \text{at} \quad \xi = 0, \tag{4.3b}$$

$$\epsilon^{1/2}c\frac{1}{\rho_\xi}\phi_\xi = -(1-\epsilon h)(ck^2+\alpha)\phi \quad \text{at} \quad \xi = -1. \tag{4.3c}$$

In terms of the new variables, (3.12) becomes

$$(1-\epsilon h)\, c(ck^2+\alpha)\phi(-1)+h\int_{-1}^{0}[k^2(\epsilon^{1/2}c-u)^2+\epsilon^{1/2}\alpha(\epsilon^{1/2}c-u)]\,\phi\,\mathrm{d}\xi = 0. \tag{4.4}$$

First, we observe that, to the leading order, the boundary-value problem (4.3) seems to have no solutions at all. Indeed, omitting small terms in (4.3), we obtain an incompatible system:

$$\left(u^2\frac{1}{\rho_\xi}\phi_\xi\right)_\xi = 0, \tag{4.5}$$

$$\phi_\xi = 0 \quad \text{at} \quad \xi = 0, \tag{4.6}$$

$$\phi = 0 \quad \text{at} \quad \xi = -1. \tag{4.7}$$

In order to resolve the paradox, we note that the leading-order equation (4.5) is inapplicable in the vicinity of $\xi = -1$, where $u(\xi)\to 0$ and the (omitted) term $\epsilon^{1/2}c$ is comparable to $u(\xi)$. Accordingly, there is a boundary layer located at $\xi = -1$, and the corresponding boundary condition (4.7) must be dropped. The remaining equations (4.5)–(4.6) describe the 'outer' solution and can be readily solved:

$$\phi_{out} = 1. \tag{4.8}$$

The thickness of the boundary layer can be determined via comparison of $\epsilon^{1/2}c$ to $u(\xi)$. Given that $u_\xi(-1) = O(1)$, the inner variable should be scaled as follows:

$$\zeta = (\xi+1)/\epsilon^{1/2}. \tag{4.9}$$

Substituting (4.9) into (4.3a, c) and omitting small terms, we get

$$\left.\begin{aligned} &\left[(c-u'\zeta)^2\frac{1}{\rho'}\phi_\zeta\right]_z = 0, \\ &c\frac{1}{\rho'}\phi_\zeta = -(ck^2+\alpha)\phi \quad \text{at} \quad \zeta = 0, \end{aligned}\right\} \tag{4.10}$$

where $u' = u_\xi(-1)$, $\rho' = \rho_\xi(-1)$. The solution to (4.10) is

$$\phi_{in} = A+B/(c-u'\zeta), \tag{4.11a}$$

where the constants of integration A and B satisfy the condition

$$B = -\frac{c(ck^2+\alpha)}{ck^2+\alpha+s}A \tag{4.11 b}$$

(in deriving (4.11 *b*), we used (3.8 *c*)).

In order to match ϕ_{in} to ϕ_{out}, we take the limit $\zeta \to \infty$ in (4.11) and compare it to (4.8), which yields

$$A = 1. \tag{4.11 c}$$

In principle, the dispersion relation $c(k)$ can be determined from the next approximation for ϕ_{out} (which will be done for the case of the continuous model in Appendix A). In the case of the mixed model, however, it can be found in a simpler way through equality (4.4). Substituting ϕ_{in} into the first term and ϕ_{out} into the second term (the contribution of the boundary layer into the integral is insignificant), we obtain (to the leading order)

$$c(ck^2+\alpha)\frac{s}{ck^2+\alpha+s}+k^2E = 0, \tag{4.12}$$

where

$$E = h\int_{-1}^{0} u^2\,\mathrm{d}\xi.$$

The solution to (4.12) can be written in the 'two-layer' form (2.12) with

$$\bar{u} = -s, \quad \bar{h} = E/s^2. \tag{4.13 a, b}$$

As before, $\bar{u}$ and $\bar{h}$ are the effective velocity and depth of the upper layer (observe that $\bar{h}$ does not have to coincide with h). The stability criterion, in this case, restricts the slope of the interface between the active and passive layers:

$$-\alpha \leqslant s \leqslant 0. \tag{4.14}$$

Evidently, criterion (4.14) coincides with (3.18) in the limit $h \to 0$. It should be noted, however, that (3.18) was derived as a sufficient condition, while (4.14) is the sufficient and necessary criterion.

Discussion

(i) The above analysis generalizes the corresponding three-layer results for the first two modes $c_{1,2}$, while the third mode c_3 does not seem to have an analogue in the mixed model at all. Straightforward asymptotic analysis demonstrates that the mixed model may have eigenvalues $O(\epsilon)$ (similar to c_3) only if the active layer includes a 'sublayer', where the velocity is almost constant: $u(z) \approx u_0$. It can be derived further that, in this case, $c \approx u_0$. (Rigorously speaking, these conclusions apply only to the regime of strong β-effect and thin upper layer, and only to medium disturbances.)

(ii) It should be noted that, in contrast to the boundary-value problem for waves in still water, the boundary-value problem (3.11) was found to have a finite number of modes. This result should not come as a surprise, as critical levels in continuously sheared currents are known to reduce the number of normal modes in similar problems (e.g. Dikiy 1976). It is also a possibility that the modes $c_{1,2}(k)$ represent, in fact, 'bunches' of closely located dispersion curves, which our asymptotic method is unable to resolve.

(iii) We emphasize, however, that, apart from the (possible) elimination of baroclinic modes, the influence of critical levels is negligible. Indeed, in the unstable case $\operatorname{Im} c \neq 0$ and critical levels simply do not occur ($\epsilon^{1/2}c-u(\xi)$ never vanishes). In the stable case, critical levels can occur only inside the boundary layer, where $u(\xi)$ is small and can be matched by $\epsilon^{1/2}c$. Strictly speaking, in this case we should consider one more (smaller) boundary layer inside the old one. There are two possibilities: the critical level can eliminate the mode altogether; or it can shift c slightly into the complex region (which would mean weak instability). In either case, this possible higher-order instability will be neglected.

(iv) Dispersion relations (2.12), (4.13) can also be used to estimate $c_{1,2}$ in the case of short disturbances. Although these formulae are valid up to, but not including, the short-wave range (2.7*c*), they can provide an estimate for $c_{1,2}$ by the order of magnitude (in any case, this estimate will be verified later by direct substitution into the equations).

Returning to the non-scaled variables (i.e. reversing formulae (4.1)–(4.2)) we then substitute $h \sim s \sim v \sim \epsilon$, $k^2 \sim \epsilon^{-1}$ into (4.13) and (2.12):

$$c_1 \rightarrow -(\alpha - \bar{u})/k^2 = O(\epsilon^2), \tag{4.15a}$$

$$c_2 \rightarrow \bar{h}\bar{u}k^2 = O(\epsilon). \tag{4.15b}$$

These estimates will be used in the next section.

5. Mixed model: short disturbances

The wavenumber of a short disturbance is to be scaled exactly as it was in the case of three-layer model:

$$k = \epsilon^{-1/2}\hat{k}. \tag{5.1}$$

c_1 will be considered first.

5.1. *The first mode*

According to (4.15*a*), c is to be scaled as follows:

$$c = \epsilon^2 \hat{c}. \tag{5.2}$$

Intending to simplify the original eigenvalue problem, we substitute (4.1), (5.1)–(5.2) into (3.11) and omit small terms and hats:

$$\left(u^2 \frac{1}{\rho_\xi}\phi_\xi\right)_x + hk^2u^2\phi = 0, \tag{5.3}$$

$$\phi_\xi = 0 \quad \text{at} \quad \xi = 0, \tag{5.4}$$

$$\phi = 0 \quad \text{at} \quad \xi = -1. \tag{5.5}$$

Similarly to the case of medium disturbances, the lower boundary condition (5.5) is incompatible with equation (5.3) and should be dropped. In order to derive the correct boundary condition, we first write it in the general form

$$\phi = a\phi^{(1)} + b\phi^{(2)}, \tag{5.6a}$$

where the two linearly independent solutions $\phi^{(1,2)}$ are fixed by their asymptotics:

$$\left.\begin{aligned} &\phi^{(1)} \rightarrow 1 + O(\xi+1), \\ &\phi^{(2)} \rightarrow \frac{1}{u'(\xi+1)} + \frac{u''}{u'}\ln(\xi+1) + O[(\xi+1)^2\ln(\xi+1)], \end{aligned}\right\} \quad \text{as} \quad \xi \rightarrow -1; \tag{5.6b}$$

and $u' = u_\xi(-1)$, $u'' = u_{\xi\xi}(-1)$. As before, one of the coefficients may be equated to unity:

$$a = 1, \tag{5.7}$$

whereas b must be found via consideration of the boundary layer of thickness $\sim \epsilon$ in the vicinity of $\xi = -1$. Omitting the calculations (which are very similar to the case of medium disturbances), we obtain

$$\phi_{in} = A + B/(c - u'\zeta), \tag{5.8a}$$

where
$$B = -c(ck^2+\alpha)A/(ck^2+\alpha+s), \tag{5.8b}$$
$$\zeta = (\xi+1)/\epsilon \tag{5.9}$$

(compare (5.8) to (4.11) and (5.9) to (4.9)). Matching (5.6) to (5.8*a*), we obtain
$$A = a, \quad -\epsilon B/u' = b,$$
which, together with (5.7) and (5.8*b*), yields
$$b = \frac{\epsilon}{u'}\frac{c(ck^2+\alpha)}{ck^2+\alpha+s}. \tag{5.10}$$

Although b seems to be small ($\sim \epsilon$), it may not be dropped from the boundary condition (5.6*a*) – otherwise the whole eigenvalue problem becomes independent of c and hence inconsistent. In order to prove the importance of b, we introduce a new eigenvalue c_{new}:
$$c = -(\alpha+s/k^2)+\epsilon c_{new}. \tag{5.11}$$

Substituting (5.11) into (5.10) and omitting small terms, we obtain an order-one expression:
$$b = -(\alpha+s)s/(u'k^4 c_{new}). \tag{5.12}$$

Equations (5.3)–(5.4), (5.6)–(5.7) and (5.12) form a boundary-value problem for c_{new}. It can be demonstrated (see Appendix B) that it has only real eigenvalues. Thus, the first mode is stable.

5.2. *The second mode*

Scaling c in accordance with (4.15*b*):
$$c = \epsilon\hat{c}, \tag{5.13}$$
we substitute (4.1), (5.1) and (5.13) into the stability boundary-value problem (3.11) and omit small terms and hats:
$$\left[(c-u)^2\frac{1}{\rho_\xi}\phi_\xi\right]_\xi + hk^2(c-u)^2\phi = 0, \tag{5.14a}$$
$$\left.\begin{aligned}(c-u)\phi_\xi = -u_\xi\phi \quad &\text{at} \quad \xi = 0,\\ \phi = 0 \quad &\text{at} \quad \xi = -1.\end{aligned}\right\} \tag{5.14b}$$

A sufficient stability condition for (5.14) can be derived similarly to the case of the exact boundary-value problem (3.9). Moreover, it just follows from the exact stability criterion (3.14) and the assumption that $((1/\rho_\xi)u_\xi)_\xi \gg \alpha$:
$$\left.\begin{aligned}&\left(\frac{1}{\rho_\xi}u_\xi\right)_\xi \quad \text{does not change sign,}\\ &u_\xi(0) \quad \text{has the same sign as} \quad \left(\frac{1}{\rho_\xi}u_\xi\right)_\xi.\end{aligned}\right\} \tag{5.15}$$

6. The subtropical and subarctic frontal currents in the Northern Pacific

In this section, part of the above stability analysis (medium disturbances) will be applied to the real ocean. With regard to short disturbances, the stability criterion (5.15) requires detailed knowledge of the vertical structure of the flow, which is not available in the literature. However, short-wave instability (if any) seems to be less important: it can be conjectured that short disturbances are unable to break the flow up and can cause only a steady loss of energy.

	SA	ST_1	ST_2	ST_3
$2L$ (km)	200	210	120	150
U_{max} (m s^{-1})	0.40	0.20	−0.15	0.45
$\bar{U}$ (m s^{-1})	0.20	0.12	−0.09	0.25
$\delta\rho/\rho_0 \times 10^3$	1.3	1.3	1.3	1.8
H_a (m)	500	350	350	500
δH_a (m)	300	140	60	140
H_0 (m)	5500	5500	5500	5500

TABLE 1. Parameters of the subarctic and subtropical frontal currents in the Northern Pacific. $2L$ is the width of the flow; U_{max} the maximum velocity scale; $\bar{U}$ the average velocity in the upper layer; $\delta\rho/\rho_s$ the average density variation; H_a the depth of the active layer; δH_a the displacement of the interface between the active and passive layers; H_0 the total depth of the ocean

	SA	ST_1	ST_2	ST_3
δ	0.091	0.064	0.064	0.091
Ro	0.021	0.016	0.021	0.050
α	0.015	0.031	0.034	0.044
s	−0.024	−0.014	0.011	−0.025

TABLE 2. Parameters of the subarctic and subtropical frontal currents in the Northern Pacific: $Ro = \bar{U}/fL$ is the Rossby number; α the β-effect number (2.2); $\delta = H_a/H_0$ the relative depth of the active layer; $s = (\delta H_a R_0)/(2LH_0)$ the non-dimensional slope of the interface (R_0 is given by (2.1))

	SA	ST_2
λ_1 (km)	120	110
λ_2 (km)	235	185
λ_{max} (km)	150	130
τ (days)	16	22

TABLE 3. Parameters of baroclinic instability of the subarctic frontal current and the middle (westward) jet of the subtropical frontal current: $\lambda_{1,2}$ are the half-lengths of marginally stable disturbances; λ_{max} the half-length of the fastest growing disturbance; τ the time of the fastest growth

We shall consider the subarctic and subtropical frontal currents in the Northern Pacific. According to Roden's (1976) experimental data, the latter flow consists of two eastward jets (axes located at 27° 30′ N and 31° 30′ N) and a weaker westward jet in between (see figure 9 of Roden's paper). In what follows, we shall use the following notation: SA = subarctic frontal current; ST_1 = subtropical frontal current, northern (eastward) jet; ST_2 = subtropical frontal current, middle (westward) jet; ST_3 = subtropical frontal current, southern (eastward) jet. The estimates of the parameters of the jets are given in table 1. It should be noted that the parameters in table 1 were chosen to approximately satisfy the geostrophic balance:

$$\bar{U} = g\,\delta\rho/\rho_s\,\delta H_a/(fL).$$

Table 1 demonstrates that all four jets can be treated as flows with thin upper layer and strong β-effect ($\delta \sim \alpha \sim Ro$).

Using table 1, one can estimate the non-dimensional parameters of the jets. These are given in table 2 which shows that ST_1 and ST_2 satisfy the stability criterion (4.14) (mixed model), while ST_3 and SA are unstable. Using formulae (2.12), (4.13) (medium disturbances) with $E = \bar{h}\bar{u}^2$, we obtain table 3 which indicates that

(i) parameters of the 'medium-wave' instability correspond to parameters of mesoscale eddies in the ocean;

(ii) wavelengths of unstable perturbations are comparable to the width of the mean flow, hence we should take into account (*a*) horizontal shear and (*b*) finite displacement of isopycnal surfaces.

Conclusion (ii)(*a*) agrees with a similar prediction by Killworth (1980). (ii)(*b*), in turn, means that we should modify the large-amplitude long-wave results of Benilov (1995) for medium disturbances.

Modifications (ii)(*a*, *b*) can be rather complicated technically, but are unlikely to affect the reduction of continuously stratified, localized flows to two-layer flows. One should be encouraged by the fact that similar reductions have already been observed for long disturbances and various regimes of large-amplitude flows with horizontal shear (Benilov 1993, 1994, 1995).

Finally, we note that table 2 demonstrates that all three currents in it can be treated as flows with a thin upper layer and strong β-effect ($\delta \sim \alpha \sim Ro$). We also mention that the estimates of Benilov & Reznik (1995) indicate that the Kuroshio and Oyashio frontal currents correspond to this regime as well.

7. Conclusions

In this paper, we have examined the baroclinic instability of stratified flows localized in a thin layer. Three models have been considered:

(i) the three-layer model with two thin layers adjacent to the surface (§2);

(ii) the 'mixed' model, which consists of a thin continuously stratified layer on top of a thick homogeneous layer (§§3–5);

(iii) the continuous model with stratification and flow localized in a thin upper layer, i.e. the general case Appendix A.

The extent of vertical localization of the flow is characterized by the parameter

$$\epsilon = H_a/H_0 \ll 1, \tag{7.1}$$

where H_a is the depth of the active layer (i.e. the layer where the flow and stratification are localized), and H_0 is the total depth of the fluid. Apart from this, we used two additional assumptions:

$$\alpha \sim \epsilon, \quad Ro \sim \epsilon; \tag{7.2}$$

where α characterizes the β-effect (see (2.2)) and Ro is the Rossby number. It should be emphasized, however, that (7.2) can be replaced by any pair of conditions which guarantee the smallness of α and Ro:

$$\alpha = o(1), \quad Ro = o(1).$$

Thus, the only vital restriction of the results obtained is (7.1) (which, however eliminates from our consideration the Gulf Stream and ACC). It is also worth noting that (7.2) corresponds to the regime of strong β-effect and thin upper layer (see Benilov & Reznik 1995) and applies to a number of currents in the Northern Pacific.

It was demonstrated that the stability of disturbances in a localized flow strongly depends on the wavelength λ. Three spectral ranges can be distinguished:

$$\begin{aligned} \text{long disturbances:} \quad & \lambda^2 \gtrsim R_0^2, \\ \text{medium disturbances:} \quad & R_a^2 \ll \lambda^2 \ll R_0^2, \\ \text{short disturbances:} \quad & \lambda^2 \lesssim R_a^2, \end{aligned}$$

where R_0 is the deformation radius based on the total depth of the fluid (see (2.1)), and R_a is based on the depth of the active layer (see (2.19)).

Equivalent two-layer model	=	Models with profiled active layer
$(\rho_2-\rho_1)/\rho_1$	=	$\delta\rho/\rho_s$
H_0	=	H_0
$\bar{h}_1$	=	$f^2E/(g'^2s^2)$
$\bar{u}_1$	=	$-(g'/f)s$

TABLE 4. Relationships between the (dimensional non-scaled) parameters of the equivalent two-layer model and model with profiled active layer: H_0 is the total depth of the fluid; $\rho_s = \rho(0)$, $\delta\rho = \rho(-H_0)-\rho(0)$; $u(z)$ is the velocity; $E = \int_{-H_0}^{0} u^2\,\mathrm{d}z$; s is the slope of the interface between the active and passive layers. Effective parameters of the equivalent two-layer model are marked with overbars; and it is assumed that $\bar{u}_2 = u(-H_0) = 0$, i.e. the passive layer is at rest for both two-layer and profiled models

(i) The stability of long disturbances cannot be studied within the framework of the traditional quasi-geostrophic equations, as their wavelengths are comparable to the 'outcropping distance' of the mean flow (i.e. the distance over which the displacement of isopycnal surfaces becomes comparable to the depth of the active layer). The question of the stability of long disturbances is addressed by Benilov (1995) using the approximation of large-amplitude geostrophic flows.

(ii) It has been demonstrated (§§2–4) that medium disturbances are described by the equivalent two-layer model, the parameters of which are related to the parameters of the three-layer and mixed models as shown in table 4 (in this section we use dimensional non-scaled variables, but keep the same notation as used earlier for the non-dimensional scaled variables, i.e. without tildas). The criterion of stability for the three-layer and mixed models follows from the corresponding two-layer condition and restricts the slope of the interface:

$$-\beta H_0/f \leqslant s \leqslant 0, \tag{7.3}$$

where $\beta = (f/R_d)\,\alpha$ is the standard β-parameter. Table 4 and condition (7.3) suggest that the crucial characteristic of a model with a 'profiled' active layer is the effective slope s of the interface between the active and passive layers. For the three-layer and mixed models s is given by

$$s = \frac{f\rho_1}{g}\frac{1}{\rho_3-\rho_2}(u_2-u_3), \quad s = \frac{f\rho_s}{g}\left(\frac{1}{\rho_z}u_z\right)_{z=H_a}, \tag{7.4}$$

respectively (here H_a is the depth of the continuously stratified layer within the framework of the mixed model).

In the (general) case of continuous stratification (Appendix A), we required additionally that the flow $u(z)$ decreases sufficiently fast in the passive layer. Rewriting (A 5) in dimensional form, we have

$$u(z)/u(0) \ll \epsilon^{1/2} \quad \text{for} \quad z/H_0 \sim 1$$

with ϵ defined by (7.1). Given this, the phase speed c can be found from the following dispersion equation:

$$[f+k^2E/cs(c)]\,c(ck^2+\beta)+R_0\,k^2E = 0, \tag{7.5a}$$

where

$$s(c) = -\frac{f\rho_s}{g}\left[c\int_{-H_0}^{0}\frac{\rho_z}{(c-u)^2}\mathrm{d}z\right]^{-1}, \tag{7.5b}$$

$$E = \int_{-H_0}^{0} u^2\,\mathrm{d}z.$$

If the slope of isopycnal surfaces in the passive layer is constant, the solution of (7.5) can be reduced to the two-layer form, and the continuous model is included in table 4. This conclusion also applies to the case of passive layer with *variable* slope of the isopycnal surface, provided the density variation $\delta\rho^{(p)}$ across this layer is sufficiently small:

$$\delta\rho^{(p)}/\rho_s \ll H_0\,\epsilon^{1/2}/(R_d\, s_0),$$

where s_0 is the effective slope of isopycnal surfaces at the lower boundary of the active layer (this condition is the dimensional version of (A 19)).

Using the above results, we estimated the maximum growth rate and spatial scale of the baroclinic instability of the subarctic and subtropical frontal currents in the Northern Pacific (§6). It has been demonstrated that the parameters of unstable medium-wave disturbances correspond to mesoscale vortices in the ocean.

(iii) In the case of short disturbances (§5 for the mixed model, §A.3 for the continuous model), the stability boundary-value problem cannot be solved in the general case. It is possible, however, to derive a sufficient condition of stability:

$$\left.\begin{array}{ll} ((1/\rho_z)\,u_z)_z & \text{does not change sign,} \\ u_z(0) & \text{has the same sign as} \quad ((1/\rho_z)\,u_z)_z. \end{array}\right\} \qquad (7.6)$$

Condition (7.6) is the standard baroclinic stability criterion adapted for the regime of a thin upper layer and weak β-effect.

Finally, the results obtained in this paper could be extended along the following lines.

(i) The accuracy of the results obtained is $O(\epsilon^{1/2})$. Given that the ratio of the depth of the active layer to the total depth of the real ocean is 0.05–0.1, this needs to be improved (the asymptotic solution is compared to the exact solution in figures 2 and 3). In other words, it is necessary to calculate the next-order correction. It is also worth noting that the next correction will include other regimes of geostrophic flows (not just the regime of weak β-effect).

(ii) It has been demonstrated (§6) that y-independent models have only limited relevance to the real ocean. Thus, it is necessary to generalize our results for flows with both vertical and horizontal shear.

(iii) As the wavelengths of unstable perturbations are comparable, in some cases, to the 'outcropping distance' of the mean flow, it is necessary to give up the quasi-geostrophic governing equations and reproduce our results for primitive equations.

(iv) It seems possible to develop a similar approximation for the internal-wave stability of localized stratified flows.

Appendix A. Continuous model

This case is described by the standard boundary-value problem:

$$\left[(c-u)^2\frac{1}{\rho_z}\phi_z\right]_z + [k^2(c-u)^2+\alpha(c-u)]\,\phi = 0, \qquad (\text{A }1a)$$

$$\phi_z = 0 \quad \text{at} \quad z = 0, \qquad (\text{A }1b)$$

$$\phi_z = 0 \quad \text{at} \quad z = -1 \qquad (\text{A }1c)$$

(compare (A 1c) with the lower mixed-model boundary condition in (3.11b)). As before, our attention is focused on the regime of weak β-effect and thin upper layer:

$$\alpha = \epsilon\hat{\alpha}, \quad u = \epsilon\hat{u}(\xi), \quad \rho = \hat{\rho}(\xi), \qquad (\text{A }2)$$

where $\xi = z/\epsilon$. Medium disturbances will be considered first.

A.1. *Medium disturbances*

The wavenumber and phase speed are to be scaled similarly to the cases of the three-layer and mixed models:

$$k = \epsilon^{-1/4}\hat{k}, \quad c = \epsilon^{3/2}\hat{c}. \tag{A 3}$$

Substitution of (A 2)–(A 3) into (A 1) yields (hats omitted)

$$\left[(\epsilon^{1/2}c-u)^2\frac{1}{\rho_\xi}\phi_\xi\right]_\xi + h[\epsilon^{1/2}k^2(\epsilon^{1/2}c-u)^2+\epsilon\alpha(\epsilon^{1/2}c-\epsilon u)]\,\phi = 0, \tag{A 4a}$$

$$\phi_\xi = 0 \quad \text{at} \quad \xi = 0, \tag{A 4b}$$

$$\phi_\xi = 0 \quad \text{at} \quad \xi = -1/\epsilon. \tag{A 4c}$$

Three asymptotic zones for the boundary-value problem (A 4) can be distinguished:

active layer: $u \gg \epsilon^{1/2}$;

intermediate layer: $u \sim \epsilon^{1/2}$;

passive layer: $u \ll \epsilon^{1/2}$.

It is worth noting that the intermediate layer can be considered as a boundary layer inside the active layer, near its lower boundary (as it was in the case of the mixed model). We shall assume that $u(\xi)$ decreases sufficiently fast:

$$u = o(\xi^{1/2}) \quad \text{as} \quad \xi \to -\infty, \tag{A 5}$$

which guarantees the existence of the passive layer.

(i) *Active layer*

Omitting the bottom boundary condition (A 1*c*), we have, to the leading order,

$$\left(u^2\frac{1}{\rho_\xi}\phi_\xi\right)_\xi = 0, \quad f_x = 0 \quad \text{at} \quad \xi = 0;$$

which yields
$$\phi = \text{const}.$$

Putting const = 1 and calculating the next correction, we have

$$\phi^{(a)} \approx 1-\epsilon^{1/2}\int_\xi^0 \rho_{\xi'}(\xi')\,u^{-2}(\xi')\int_{\xi'}^0 k^2u^2(\xi'')\,\mathrm{d}\xi''\,\mathrm{d}\xi'.$$

In order to match the active layer to the intermediate layer, we shall need

$$\left.\begin{aligned}\phi^{(a)} &\approx 1\\ (\epsilon^{1/2}-u)^2\frac{1}{\rho_\xi}\phi_\xi^{(a)} &\approx \epsilon^{1/2}k^2E\end{aligned}\right\} \quad \text{as} \quad \xi \to -\infty, \tag{A 6}$$

where
$$E = \int_{-\infty}^0 u^2\,\mathrm{d}\xi. \tag{A 7}$$

Observe that the lower boundary of the active layer has now moved to $-\infty$. It is also worth noting that convergence of E is guaranteed by condition (A 5).

(ii) *Intermediate layer*

It turns out that in this case we need the leading-order solution only. Having in mind that $u \sim \epsilon^{1/2}c$, we obtain

$$[(\epsilon^{1/2}c-u)^2(1/\rho_\xi)\,\phi_\xi]_\xi = 0,$$

which yields
$$\phi^{(i)} \approx \text{const}_1 + \text{const}_2 \int_{\xi}^{0} \frac{\rho_{\xi'}(\xi')}{[\epsilon^{1/2}c - u(\xi')]^2} \mathrm{d}\xi.$$

In order to match the intermediate layer to the other two layers, we take the limits
$$\left.\begin{aligned} \phi^{(i)} &\approx \text{const}_1 \\ (\epsilon^{1/2}-u)^2 \frac{1}{\rho_\xi} \phi_\xi^{(i)} &\approx -\text{const}_2 \end{aligned}\right\} \quad \text{as} \quad \xi \to 0 \tag{A 8}$$

and
$$\left.\begin{aligned} \phi^{(i)} &\approx \text{const}_1 + \text{const}_2 \int_{-\infty}^{0} \frac{\rho_\xi(\xi)}{[\epsilon^{1/2}c - u(\xi)]^2} \mathrm{d}\xi \\ (\epsilon^{1/2}-u)^2 (1/\rho_\xi)\, \phi_\xi^{(i)} &\approx -\text{const}_2 \end{aligned}\right\} \quad \text{as} \quad \xi \to -\infty. \tag{A 9}$$

(iii) *Passive layer*

In this case $u \ll \epsilon^{1/2}c$ and the solution to (A 4*a*, *c*) is
$$\phi \approx \text{const}_3 - \int_{-1/\epsilon}^{\xi} \rho_{\xi'}(\xi') \frac{ck^2+\alpha}{c} \text{const}_3 \left(\xi' + \frac{1}{\epsilon}\right) \mathrm{d}\xi'.$$

In order to match the passive layer to the intermediate layer, we take the limit
$$\left.\begin{aligned} \phi^{(p)} &\approx \text{const}_3 \\ (\epsilon^{1/2}-u)^2 (1/\rho_\xi)\, \phi_\xi^{(p)} &\approx -\epsilon^{1/2} \text{const}_3\, c(ck^2+\alpha) \end{aligned}\right\} \quad \text{as} \quad \xi \to 0. \tag{A 10}$$

Now, we match (A 6) to (A 8) and (A 9) to (A 10). Eliminating $\text{const}_{1,2,3}$, we obtain
$$[1 + k^2 E / cs(c)]\, c(ck^2+\alpha) + k^2 E = 0, \tag{A 11}$$

where
$$s(c) = -\left[\epsilon^{1/2} c \int_{-\infty}^{0} \frac{\rho_\xi}{(\epsilon^{1/2}c - u)^2} \mathrm{d}\xi\right]^{-1}. \tag{A 12}$$

$s(c)$ can be simplified using the condition $\epsilon \ll 1$. First, we rewrite (A 12) in terms of $u(\rho)$ (which replaces $\rho(\xi)$ and $u(\xi)$):
$$s(c) = -\left[\epsilon^{1/2} c \int_{1}^{0} \frac{\mathrm{d}\rho}{[\epsilon^{1/2}c - u(\rho)]^2}\right]^{-1}, \tag{A 13}$$

then introduce s_0 such that
$$\lim_{\rho \to 1} \mathrm{d}u/\mathrm{d}\rho = s_0. \tag{A 14}$$

Equation (B 13) yields
$$s(c) \approx -\left[\epsilon^{1/2} c \int_{1}^{0} \frac{\mathrm{d}\rho}{[\epsilon^{1/2}c - s_0(\rho-1)]^2}\right]^{-1} \approx s_0. \tag{A 15}$$

Now, solving equation (A 11) with (A 15), one can see that the dispersion relation $c(k)$ can be written in the two-layer form (2.12) with
$$\bar{u} = -s_0, \quad \bar{h} = E/s_0^2. \tag{A 16a, b}$$

Moreover, the similarity of formulae (A 16) (continuous model) and (4.13) (mixed model) demonstrates that s_0 can be interpreted as the effective slope of the interface between the active and passive layers.

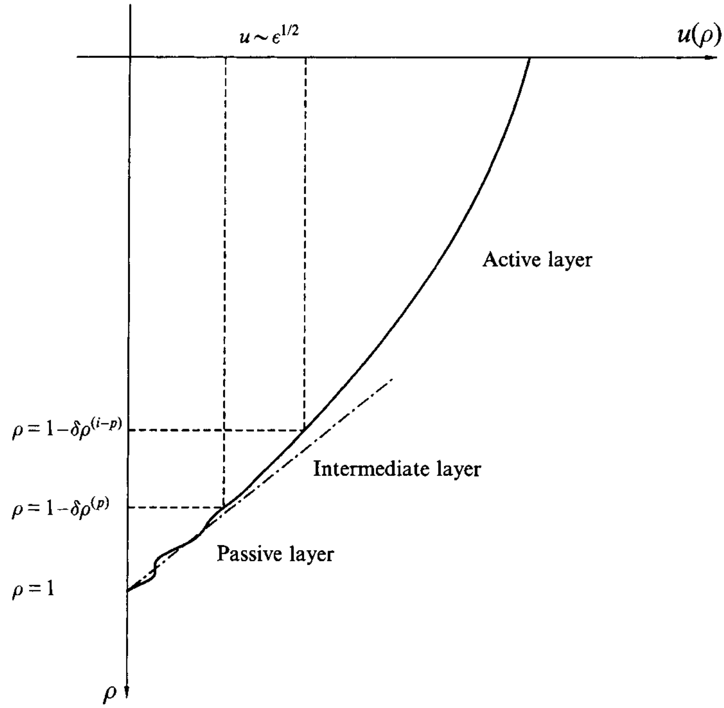

FIGURE 5. Velocity (scaled by ϵ) *vs*. density. $\epsilon = H_a/H_0 \ll 1$. Active layer: $u \gg \epsilon^{1/2}$; intermediate layer; $u \sim \epsilon^{1/2}$, $\delta\rho^{(i)}$ is the density variation across the intermediate layer; passive layer: $u \ll \epsilon^{1/2}$, $\delta\rho^{(p)}$ is the density variation across the passive layer. Observe that the passive layer may include sharp changes of the slope of isopycnal surfaces (given by $\mathrm{d}u/\mathrm{d}\rho$). Dashed-dotted line extrapolates the velocity profile of the intermediate layer across the passive layer to the point $\rho = 1$.

A.2. *Discussion*

In this section, we shall discuss how the asymptotic behaviour of $u(\rho)$ at $\rho = 1$ can affect the solution to the dispersion equation (A 11)–(A 12).

First, we note that assumption (A 14) can be rewritten as

$$\lim_{\xi \to -\infty} u_\xi/\rho_\xi = s_0,$$

and, in this form, means that the slope of isopycnal surfaces throughout the passive layer is constant, which severely restricts practical applicability of the results obtained. Of course, we can always give up assumption (A 14) and use (A 12) in its general form. In this case, however, it is not clear if (A 11)–(A 12) can be reduced to the two-layer dispersion relation.

In order to clarify this question, we note that the passive layer is represented by a narrow strip on the (u, ρ) graph with fast varying $u(\rho)$ (see figure 5). It is convenient to split $u(\rho)$ as follows:

$$u(\rho) = \begin{cases} u^{(a)}(\rho) & \text{if} \quad 0 \leqslant \rho \leqslant 1 - \delta\rho^{(i-p)}, \\ u^{(i)}(\rho) & \text{if} \quad 1 - \delta\rho^{(i-p)} \leqslant \rho \leqslant 1 - \delta\rho^{(p)}, \\ u^{(p)}(\rho) & \text{if} \quad 1 - \delta\rho^{(p)} \leqslant \rho \leqslant 1, \end{cases} \tag{A 17}$$

where $u^{(a)} \sim 1$ represents the active layer, $u^{(i)} \sim \epsilon^{1/2}$ represents the intermediate layer, and $u^{(p)} \ll \epsilon^{1/2}$ represents the passive layer. $\delta\rho^{(p)}$ and $[\delta\rho^{(i-p)} - \delta\rho^{(p)}]$ represent the density variations across the passive and intermediate layers, respectively.

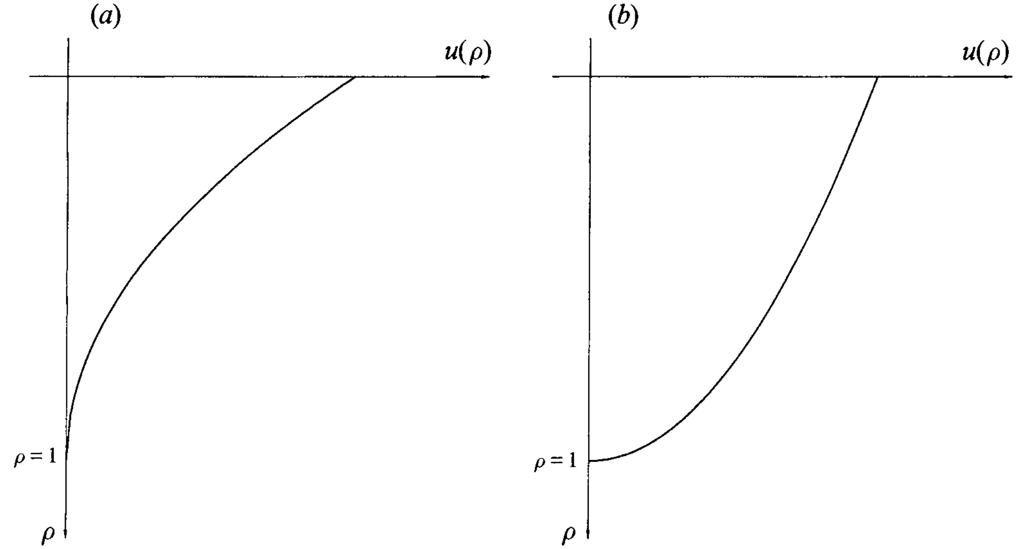

FIGURE 6. Velocity (scaled by ϵ) *vs.* density. (*a*) Zero slope of the interface between the active and passive layers: $du/d\rho = 0$ at $\rho = 1$. (*b*) Infinite slope of the interface between the active and passive layers: $du/d\rho = \infty$ at $\rho = 1$.

Evidently, the contribution of the active layer to the integral in (A 13) is negligible (the denominator of the integrand is too big there). The intermediate layer does contribute to the integral, whereas the passive layer contributes only if its width is not much smaller than that of the intermediate layer. In order to find the width of the intermediate layer, we extrapolate $u^{(i)}(\rho)$ across the passive layer to $\rho = 1$ (see figure 5, the dash-dotted line) and introduce

$$s_0 = du_{ext}^{(i)}/d\rho|_{\rho=1}. \tag{A 18}$$

Recalling that the velocity change across the intermediate and passive layers is $O(\epsilon^{1/2})$, we obtain

$$\delta\rho^{(i-p)} \sim \epsilon^{1/2}/s_0. \tag{A 19a}$$

Clearly, if

$$\delta\rho^{(p)} \ll \delta\rho^{(i-p)}, \tag{A 19b}$$

the passive layer does not contribute to $s(c)$, and the fast varying $u^{(p)}$ can be replaced by $u_{ext}^{(i)}$ extrapolated from the intermediate layer. As a result, $s(c)$ can be calculated exactly as it was in (A 15):

$$s(c) \approx s_0, \tag{A 20}$$

and the dispersion equations (A 11), (A 20) take the two-layer form. Unlike $\delta\rho^{(i-p)}$, the width $\delta\rho^{(p)}$ of the passive layer cannot be calculated: it just represents the density variation across the layer with sharp changes of the slope of isopycnal surfaces.

In practical use, however, it is difficult to precisely separate the intermediate and passive layers (as we did theoretically in (A 17)) and hence determine s_0. The only way to avoid arbitrariness seems to be to use the exact expression (A 12), rather than its approximate form (A 20), (A 18). If solved with the dispersion equation (A 11), the exact expression will 'choose' the correct value of s_0 naturally.

Another important question is what happens when the effective slope of the interface is zero (figure 6*a*) or infinite (figure 6*b*) – in both cases we assume, for simplicity, that there are no sharp changes of the slope of isopycnal surfaces in the passive layer. Of course, approximation (A 20), (A 18) does not work in this case; moreover, even the exact expression (A 12) is likely to be incorrect. Indeed, given that

$$u \to \text{const}\,(1-\rho)^n \quad \text{as} \quad \rho \to 1,$$

(A 12) yields

$$s = O(\epsilon^{(1-n)/2n}).$$

If $n \neq 1$, the small parameter ϵ does not disappear from the dispersion equation (A 11), which indicates that our original scaling (A 3) was incorrect. We shall not go into detail of this question, but note only that this shortcoming of the asymptotic theory presented can be readily amended for every particular value of n.

Finally, we note that the infinite lower limits in (A 7) and (A 12) could be replaced by $-1/\epsilon$. Rewriting (A 7) and (A 12) in terms of the non-scaled non-dimensional variables, we have

$$E = \int_{-1}^{0} u^2\,\mathrm{d}z, \quad s(c) = -\left[c\int_{-1}^{0} \frac{\rho_z}{(c-u)^2}\mathrm{d}z\right]^{-1}. \tag{A 21}$$

Equation (A 21) supplements the dispersion equation (A 11).

A.3. *Short disturbances*

The dynamics of short disturbances within the framework of the continuous model are similar to those for the mixed model. Omitting details, we note that the stability criterion is exactly as it was before (i.e. as given by (5.15)).

Appendix B. Proof of stability of the second mode of short disturbances

If we separate the imaginary and real parts in (5.3)–(5.4), (5.6)–(5.7) and (5.12), this set of equations splits into two independent boundary-value problems:

$$\left[u^2\frac{1}{\rho_\xi}(\operatorname{Im}\phi)_\xi\right]_\xi + hk^2u^2(\operatorname{Im}\phi) = 0,$$

$$(\operatorname{Im}\phi)_\xi = 0 \quad \text{at} \quad \xi = 0, \quad \operatorname{Im}\phi = (\operatorname{Im} b)\,\phi^{(2)};$$

$$\left[u^2\frac{1}{\rho_\xi}(\operatorname{Re}\phi)_\xi\right]_\xi + hk^2u^2(\operatorname{Re}\phi) = 0,$$

$$(\operatorname{Re}\phi)_\xi = 0 \quad \text{at} \quad \xi = 0, \quad \operatorname{Re}\phi = \phi^{(1)} + (\operatorname{Re} b)\,\phi^{(2)},$$

where $\phi^{(1)}$ and $\phi^{(2)}$ are fixed by (5.6*b*) and b includes the eigenvalue c_{new} (see (5.12)).

If $\operatorname{Im} b \neq 0$, we can introduce the new variable

$$\chi = \operatorname{Re}\phi - (\operatorname{Re} b/\operatorname{Im} b)\operatorname{Im}\phi$$

and obtain

$$\left[u^2\frac{1}{\rho_\xi}\chi_\xi\right]_\xi + hk^2u^2\chi = 0, \tag{B 1a}$$

$$\chi_\xi = 0 \quad \text{at} \quad \xi = 0, \quad \chi = \phi^{(1)}. \tag{B 1b}$$

The boundary-value problem (B 1) does not depend on the eigenvalue c_{new} and therefore can have non-trivial solutions only for isolated values of k (which may be treated as a new eigenvalue). However, it can be easily demonstrated that (B 1) has no solutions at all: multiplying (B 1*a*) by χ, integrating by parts and taking into account the boundary conditions (B 1*b*), and the conditions

$$u(-1) = 0, \quad \phi^{(1)}(-1) = 1,$$

we obtain

$$-\int_{-1}^{0} u^2\frac{1}{\rho_\xi}(\chi_\xi)^2\,\mathrm{d}\xi + hk^2\int_{-1}^{0} u^2\chi^2\,\mathrm{d}\xi = 0.$$

As $\rho_\xi < 0$, this equality may not be correct, which means that our original assumption $\operatorname{Im} b \neq 0$ was also incorrect.

Thus, the boundary-value problem (5.3)–(5.4), (5.6)–(5.7) and (5.12) can have only real eigenvalues: $\operatorname{Im} c_{new} = 0$.

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